Rate of convergence for numerical solutions to SFDEs with jumps

Jianhai Bao[1], Xuerong Mao[2], Chenggui Yuan[3*]

[1]School of Mathematics, Central South University,
Changsha, Hunan 410075, P.R.China
[2]Department of Statistics and Modelling Science,
University of Strathclyde, Glasgow G1 1XH, UK
[3]Department of Mathematics, Swansea University,
Swansea SA2 8PP, UK

Abstract

In this paper, we are interested in the numerical solutions of stochastic functional differential equations (SFDEs) with *jumps*. Under the global Lipschitz condition, we show that the pth moment convergence of the Euler-Maruyama (EM) numerical solutions to SFDEs with jumps has order $1/p$ for any $p \geq 2$. This is significantly different from the case of SFDEs without jumps where the order is $1/2$ for any $p \geq 2$. It is therefore best to use the mean-square convergence for SFDEs with jumps. Consequently, under the local Lipschitz condition, we reveal that the order of the mean-square convergence is close to $1/2$, provided that the local Lipschitz constants, valid on balls of radius j, do not grow faster than $\log j$.

Keywords: Euler-Maruyama; Local Lipschitz condition; Stochastic functional differential equation; Rate of convergence; Jump processes.
Mathematics Subject Classification (2000) 65C30, 65L20, 60H10.

1 Introduction

Recently, the theory of functional differential equations (FDEs) has received a great deal of attention. Hale and Lune [6] have studied deterministic functional differential equations (DFDEs) and their stability. For stochastic functional differential equations (SFDEs), we

* *E-mail address:* jianhaibao@yahoo.com.cn, xuerong@stams.strath.ac.uk, C.Yuan@swansea.ac.uk.

here highlight the great contribution of Kolmanovskii and Nosov [7] and Mao [8]. Kolmanovskii and Nosov [7] not only established the theory of existence and uniqueness of SFDEs but also investigated the stability and asymptotic stability of the equations, while Mao [8] studied the exponential stability of the equations.

On the other hand, stochastic differential equations (SDEs) with jumps have been widely used in many branches of science and industry, in particular, in economics, finance and engineering (see, for example, Gukhal [3], R. Cont [2], Sobczyk [11] and references therein). Since most SDEs with jumps cannot be solved explicitly, numerical methods have become essential. Under the local Lipschitz condition, Higham and Kloeden [4] showed the strong convergence and nonlinear stability for the EM numerical solutions to SDEs with jumps, while, in [5], Higham and Kloeden further revealed the strong convergence rate for the backward Euler (BE) on SDEs with jumps, provided that the drift coefficient obeys one-side Lipschitz condition and polynomial growth condition.

Returning to the SFDEs, we recalled Mao [9] developed a numerical scheme for them. Under the *local Lipschitz condition*, Mao [9] showed the strong convergence of the EM numerical solutions to SFDEs, but revealed the rate of the convergence under the *global Lipschitz condition*. But there is so far no work on numerical methods for SFDEs with jumps.

Motivated by the papers mentioned above, we are here interested in the numerical solutions to SFDEs *with jumps*. In comparison with the results obtained by Mao [9], our significant contributions are:

- Under the global Lipschitz condition, we show that the pth moment convergence of the EM numerical solutions to SFDEs with jumps has order $1/p$ for any $p \geq 2$. This is significantly different from the case of SFDEs without jumps where the order is $1/2$ for any $p \geq 2$. In practice, it is therefore best to use the mean-square convergence for SFDEs with jumps.

- Under the local Lipschitz condition, Mao [9] showed the strong convergence *without rate* of the EM numerical solutions to SFDEs without jumps. However, we shall reveal that the order of the mean-square convergence is closed to $1/2$, provided that the local Lipschitz constants, valid on balls of radius j, do not grow faster than $\log j$. More precisely, the order of the mean-square convergence is $1/(2+\epsilon)$, provided that the local Lipschitz constants do not grow faster than $(\log j)^{1/(1+\epsilon)}$.

- Some new techniques are developed to cope with the difficulty due to the jumps.

This paper is organized as follows: Section 2 gives some preliminary results, in particular, the EM numerical solutions to SFDEs with jumps are set up. In section 3, we discuss the pth moment convergence of the EM numerical solutions to SFDEs with jumps under the global Lipschitz condition. The rate of the mean-square convergence of the EM numerical solutions to SFDEs with jumps under the local Lipschitz condition is provided in Section 4. Finally, in order to make the paper self-contained, an existence-and-uniqueness result of solutions to SFDEs with jumps is provided in the Appendix.

2 Preliminaries

Throughout this paper, we let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is continuous on the right and $\mathcal{F}_0$-contains all P-zero sets). Let $|\cdot|$ denote the Euclidean norm and the matrix trace norm. Let $\tau > 0$ and $D := D([-\tau, 0]; R^n)$ denote the family of all right-continuous functions with left-hand limits φ from $[-\tau, 0]$ to R^n, and $\hat{D} := \hat{D}([-\tau, 0]; R^n)$ denote the family of all left-continuous functions with right-hand limits φ from $[-\tau, 0]$ to R^n, we will always use $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$ to denote the norm in D and $\hat{D}$ potentially involved when no confusion possibly arises. $D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $D([-\tau, 0]; R^n)$-valued random variables. For all $t \geq 0$, $x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0\}$ is regarded as a $D([-\tau, 0]; R^n)$-valued stochastic process. Let $x(t^-)$ denotes $\lim_{s\uparrow t} x(s)$ on $t \in [-\tau, T]$ and $x_{t^-} = \{x(t + \theta)^- : -\tau \leq \theta \leq 0\}$, it is easy to see $x(t^-)$ is a $\hat{D}([-\tau, 0]; R^n)$-valued stochastic process.

In this paper, we consider the following SFDE with jumps

$$dx(t) = f(x_{t^-})dt + g(x_{t^-})dB(t) + h(x_{t^-})dN(t), \qquad 0 \leq t \leq T, \tag{2.1}$$

with the initial data $x_0 = \xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. Here, $f, h : \hat{D}([-\tau, 0]; R^n) \to R^n$, $g : \hat{D}([-\tau, 0]; R^n) \to R^{n\times m}$, $B(t)$ is an m-dimensional Brownian motion and $N(t)$ is a scalar Poisson process with intensity λ. We further assume that $B(t)$ and $N(t)$ are independent. It should be pointed out that the solution of Eq. (2.1) is in $D([-\tau, 0]; R^n)$

For our purposes, we need the following assumptions which can also guarantee the existence and uniqueness of solution to (2.1) (see Appendix).

(H1) (Global Lipschitz condition) There exists a left-continuous nondecreasing function $\mu : [-\tau, 0] \to R_+$ such that, for all $\varphi, \psi \in \hat{D}([-\tau, 0]; R^n)$,

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq \int_{-\tau}^{0} |\varphi(\theta) - \psi(\theta)|^2 d\mu(\theta). \tag{2.2}$$

Remark 2.1. *For simplicity, we write $L = \mu(0) - \mu(-\tau)$, which is referred to as the global Lipschitz constant. Note from (2.2) that, for all $\varphi, \psi \in \hat{D}([-\tau, 0]; R^n)$,*

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq L\|\varphi - \psi\|^2. \tag{2.3}$$

This further implies the linear growth condition; that is, for $\varphi \in \hat{D}([-\tau, 0]; R^n)$,

$$|f(\varphi)|^2 \vee |g(\varphi)|^2 \vee |h(\varphi)|^2 \leq K(1 + \|\varphi\|^2), \tag{2.4}$$

where $K = 2(L \vee |f(0)|^2 \vee |g(0)|^2 \vee |h(0)|^2)$.

(H2) (Continuity of initial data) For $\xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; R^n), 0 \leq u \leq \tau$ and $p \geq 2$, there is a constant $\beta > 0$ such that

$$E\left(\sup_{\substack{-\tau\leq s\leq t\leq 0,\\ |t-s|\leq u}} |\xi(s) - \xi(t)|^p\right) \leq \beta u. \tag{2.5}$$

For given $T \geq 0$ and $\tau > 0$, the time-step size $\triangle \in (0,1)$ is defined by

$$\triangle = \frac{\tau}{N} = \frac{T}{M}$$

with some integers $N > \tau$ and $M > T$. The EM method applied to (2.1) produces approximations $\bar{y}(k\triangle) \approx x(k\triangle)$ by setting $\bar{y}(k\triangle) = \xi(k\triangle)$, $-N \leq k \leq 0$, and

$$\bar{y}((k+1)\triangle) = \bar{y}(k\triangle) + f(\bar{y}_{k\triangle}) \triangle + g(\bar{y}_{k\triangle}) \triangle B_k + h(\bar{y}_{k\triangle}) \triangle N_k, \tag{2.6}$$

where $\triangle B_k = B((k+1)\triangle) - B(k\triangle)$ is a Brownian increment, $\triangle N_k = N((k+1)\triangle) - N(k\triangle)$ is a Poisson increment, and $\bar{y}_{k\triangle} = \{\bar{y}_{k\triangle}(\theta) : -\tau \leq \theta \leq 0\}$ is a $D([-\tau,0];R^n)$-valued random variable defined by

$$\bar{y}_{k\triangle}(\theta) = \frac{(i+1)\triangle - \theta}{\triangle}\bar{y}((k+i)\triangle) + \frac{\theta - i\triangle}{\triangle}\bar{y}((k+i+1)\triangle) \tag{2.7}$$

for $i\triangle \leq \theta \leq (i+1)\triangle$, $i = -N, -(N-1), \cdots, -1$, where in order for $\bar{y}_{-\triangle}$ to be well defined, we set $\bar{y}(-(N+1)\triangle) = \xi(-N\triangle)$.

Given the discrete-time approximation $\{\bar{y}(k\triangle)\}_{k\geq 0}$, we define a continuous-time approximation $y(t)$ by $y(t) = \xi(t)$ for $-\tau \leq t \leq 0$, while for $t \in [0,T]$,

$$y(t) = \xi(0) + \int_0^t f(\bar{y}_{s^-})ds + \int_0^t g(\bar{y}_{s^-})dB(s) + \int_0^t h(\bar{y}_{s^-})dN(s), \tag{2.8}$$

where, for fixed $\theta \in [-\tau, 0]$,

$$\bar{y}_{t^-} = \lim_{s\uparrow t} \bar{y}_s, \qquad \bar{y}_t = \sum_{k=0}^{M-1} \bar{y}_{k\triangle} I_{[k\triangle,(k+1)\triangle)}(t).$$

It is easy to see $y(k\triangle) = \bar{y}(k\triangle)$ for $k = -N, -N+1, \cdots, M$. That is, the discrete-time and continuous-time EM numerical solutions coincide at the gridpoints.

Remark 2.2. *It is easy to observe from* (2.7) *that*

$$\|\bar{y}_{k\triangle}\| = \max_{-N\leq i\leq 0} |\bar{y}((k+i)\triangle)|, \quad k = -1, 0, 1, \cdots, M-1, \tag{2.9}$$

which further yields

$$\|\bar{y}_{k\triangle}\| \leq \|y_{k\triangle}\|, \quad k = -1, 0, 1, \cdots, M-1,$$

by $y(k\triangle) = \bar{y}(k\triangle)$ *and, for any* $t \in [0,T]$,

$$\|\bar{y}_t\| = \|\bar{y}_{[\frac{t}{\triangle}]\triangle}\| \leq \|y_{[\frac{t}{\triangle}]\triangle}\| \leq \sup_{-\tau\leq s\leq t} |y(s)|, \tag{2.10}$$

where $[\frac{t}{\triangle}]$ *is the integer part of* $\frac{t}{\triangle}$.

3 Convergence under the global Lipschitz condition

In this section, we will investigate the rate of the convergence under the global Lipschitz condition. Our results reveal a significant difference from these on the SDEs without jumps.

Lemma 3.1. *Under the condition (2.4), for any $p \geq 2$ there exists a positive constant $H(p) := H(p, T, \xi, K)$ which may dependent on p, T, ξ, K such that*

$$E\Big(\sup_{-\tau \leq t \leq T} |x(t)|^p\Big) \vee E\left(\sup_{-\tau \leq t \leq T} |y(t)|^p\right) \leq H(p). \tag{3.1}$$

Proof. Since the arguments of the moment bounds for the exact and continuous approximate solutions to (2.1) are very similar, we here only give an estimate for the continuous approximate solution $y(t)$. For every integer $R \geq 1$, define the stopping time

$$\theta_R = \inf\{t \geq 0 : \|y_t\| \geq R\}$$

It is easy to see from (2.8) that, for any $t \in [0, T]$,

$$\begin{aligned}
&E\left(\sup_{0 \leq s \leq t} |y(s \wedge \theta_R)^-|^p\right) \leq E\left(\sup_{0 \leq s \leq t} |y(s \wedge \theta_R)|^p\right) \\
&\leq 4^{p-1}\Big[E\|\xi\|^p + E\left(\sup_{0 \leq s \leq t}\left|\int_0^s f(\bar{y}_{(r\wedge\theta_R)^-})dr\right|^p\right) \\
&+ E\left(\sup_{0 \leq s \leq t}\left|\int_0^s g(\bar{y}_{(r\wedge\theta_R)^-})dB(r)\right|^p\right) + E\left(\sup_{0 \leq s \leq t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})dN(r)\right|^p\right)\Big].
\end{aligned} \tag{3.2}$$

Noting that $E\|y_{(t\wedge\theta_R)^-}\| \leq R$ and (2.10), one may have $E\|\bar{y}_{(t\wedge\theta_R)^-}\| \leq R$. By the Hölder inequality and (2.4),

$$\begin{aligned}
E\left(\sup_{0 \leq s \leq t}\left|\int_0^s f(\bar{y}_{(r\wedge\theta_R)^-})dr\right|^p\right) &\leq T^{p-1}\int_0^t E|f(\bar{y}_{(r\wedge\theta_R)^-})|^p dr \\
&\leq T^{p-1}\int_0^t E[K(1 + \|\bar{y}_{(r\wedge\theta_R)^-}\|^2)]^{\frac{p}{2}} dr \\
&= T^{p-1}\int_0^t E[K(1 + \|\bar{y}_{(r\wedge\theta_R)^-}\|^2)]^{\frac{p}{2}} dr \\
&\leq 2^{\frac{p}{2}-1}T^p K^{\frac{p}{2}} + 2^{\frac{p}{2}-1}T^{p-1}K^{\frac{p}{2}}\int_0^t E\|\bar{y}_{(r\wedge\theta_R)^-}\|^p dr.
\end{aligned}$$

This, together with (2.10), immediately reveals that

$$E\left(\sup_{0 \leq s \leq t}\left|\int_0^s f(\bar{y}_{(r\wedge\theta_R)^-})dr\right|^p\right) \leq c_1 T + c_1\int_0^t E\left(\sup_{-\tau \leq r \leq s} |y(r \wedge \theta_R)^-|^p\right) ds, \tag{3.3}$$

where $c_1 = 2^{\frac{p}{2}-1}T^{p-1}K^{\frac{p}{2}}$. Now, using the Burkholder-Davis-Gundy inequality [8, Theorem 7.3, p40] and the Hölder inequality, we deduce that there exists a positive constant c_p such

that

$$E\left(\sup_{0\le s\le t}\left|\int_0^s g(\bar{y}_{(r\wedge\theta_R)^-})dB(r)\right|^p\right)\le c_pE\left(\int_0^t |g(\bar{y}_{(r\wedge\theta_R)^-})|^2dr\right)^{p/2}$$
$$\le c_pT^{\frac{p-2}{2}}\int_0^t E|g(\bar{y}_{r\wedge\theta_R})^-|^pdr.$$

In the same way as (3.3) was done, it then follows easily that

$$E\left(\sup_{0\le s\le t}\left|\int_0^s g(\bar{y}_{(r\wedge\theta_R)^-})dB(r)\right|^p\right)\le c_2T+c_2\int_0^t E\left(\sup_{-\tau\le r\le s}|y(r\wedge\theta_R)^-|^p\right)ds,$$

where $c_2=2^{\frac{p}{2}-1}T^{\frac{p-2}{2}}K^{\frac{p}{2}}c_p$. Moreover, observing that $\tilde{N}(t)=N(t)-\lambda t, t\ge 0$ is a martingale measure, using the Burkholder-Davis-Gundy inequality [10, Theorem 48, p193], Hölder inequality and (2.4), we obtain for some positive constant $\bar{c}_p$,

$$\begin{aligned}
&E\left(\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})dN(r)\right|^p\right)\\
&\le E\left(\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})d\tilde{N}(r)+\lambda\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})dr\right|^p\right)\\
&\le 2^p\left[E\left(\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})d\tilde{N}(r)\right|^p+\lambda^p\sup_{0\le s\le t}\left|\int_0^s h(\bar{y}_{(r\wedge\theta_R)^-})dr\right|^p\right)\right]\\
&\le 2^p\left[\bar{c}_p\lambda^{p/2}E\left(\int_0^t |h(\bar{y}_{(r\wedge\theta_R)^-}))|^2dr\right)^{p/2}+\lambda^pT^{p-1}\int_0^t E|h(\bar{y}_{(r\wedge\theta_R)^-}))|^pdr\right]\\
&\le c_3T+c_3\int_0^t E\left(\sup_{-\tau\le r\le s}|y(r\wedge\theta_R)^-|^p\right)ds,
\end{aligned}$$

where $c_3=2^{\frac{3p}{2}-1}K^{\frac{p}{2}}\left[\bar{c}_p\lambda^{p/2}T^{\frac{p-2}{2}}+\lambda^pT^{p-1}\right]$. Hence, in (3.2)

$$\begin{aligned}
&E\left(\sup_{0\le s\le t}|y(s\wedge\theta_R)^-|^p\right)\\
&\le 4^{p-1}\left[E\|\xi\|^p+(c_1+c_2+c_3)T+(c_1+c_2+c_3)\int_0^t E\left(\sup_{-\tau\le r\le s}|y(r\wedge\theta_R)^-|^p\right)ds\right].
\end{aligned}$$

Note that

$$E\left(\sup_{-\tau\le s\le t}|y(s\wedge\theta_R)^-|^p\right)\le E\|\xi\|^p+E\left(\sup_{0\le s\le t}|y(s\wedge\theta_R)^-|^p\right).$$

Applying the Gronwall inequality and letting $R\to\infty$, we then obtain

$$E\left(\sup_{-\tau\le t\le T}|y(t^-)|^p\right)\le H(p).$$

Since T is any fixed positive number, the required assertion follows. □

In order to obtain our main results, we need to estimate the pth moment of $y(s+\theta)-\bar{y}_s(\theta)$.

Lemma 3.2. *Let the conditions (2.4) and (2.5) hold. Then, for $p \geq 2$ and $s \in [0, T]$,*

$$E|y(s+\theta) - \bar{y}_s(\theta)|^p \leq \gamma \triangle, \qquad -\tau \leq \theta \leq 0, \tag{3.4}$$

where γ is a positive constant which is independent of $\triangle$.

Proof. Fix $s \in [0, T]$ and $\theta \in [-\tau, 0]$. Let $k_s \in \{0, 1, 2, \cdots, M-1\}$, $k_\theta \in \{-N, -N+1, \cdots, -1\}$ be the integers for which $s \in [k_s\triangle, (k_s+1)\triangle)$, $\theta \in [k_\theta\triangle, (k_\theta+1)\triangle)$, respectively. For convenience, we write $v = s + \theta$ and $k_v = k_s + k_\theta$. Clearly, $0 \leq s - k_s\triangle < \triangle$ and $0 \leq \theta - k_\theta\triangle \leq \triangle$, so

$$0 \leq v - k_v\triangle < 2\,\triangle\,.$$

Recalling the definition of $\bar{y}_s, s \in [0, T]$, we then yield from (2.7) that

$$\bar{y}_s(\theta) = \bar{y}_{k_s\triangle}(\theta) = \bar{y}(k_v\triangle) + \frac{\theta - k_\theta\triangle}{\triangle}[\bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)],$$

which implies

$$E|y(s+\theta) - \bar{y}_s(\theta)|^p \leq 2^{p-1}E|\bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)|^p + 2^{p-1}E|y(v) - \bar{y}(k_v\triangle)|^p. \tag{3.5}$$

For $k_v \leq -1$, it thus follows from (2.5) that

$$E|\bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)|^p \leq \beta\,\triangle\,. \tag{3.6}$$

Note that, for some $\bar{H} := \bar{H}(m, p)$,

$$E|B(t)|^p \leq \bar{H}t^{\frac{p}{2}}, \qquad t \geq 0 \tag{3.7}$$

and, by the characteristic functions argument, for $\triangle \in (0, 1)$,

$$E|\triangle N_k|^p \leq C\triangle, \tag{3.8}$$

where C is a positive constant which is independent of $\triangle$. For $k_v \geq 0$, using (2.6) and noting $g(\bar{y}_{k_v\triangle})$ and B_{k_v}, $h(\bar{y}_{k_v\triangle})$ and N_{k_v} are independent, respectively, we compute

$$\begin{aligned} &E|\bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)|^p \\ &\quad \leq 3^{p-1}\left[E|f(\bar{y}_{k_v\triangle})|^p \triangle^p + E|g(\bar{y}_{k_v\triangle})|^p E|\triangle B_{k_v}|^p + E|h(\bar{y}_{k_v\triangle})|^p E|\triangle N_{k_v}|^p\right]. \end{aligned}$$

Taking (2.4) into consideration and applying Lemma 3.1, we then obtain that for $\triangle \in (0, 1)$

$$E|\bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)|^p \leq 3^{p-1}2^{\frac{p}{2}-1}K^{\frac{p}{2}}(1 + H(p))(1 + \bar{H} + C)\,\triangle\,. \tag{3.9}$$

Hence, in (3.5)

$$\begin{aligned} E|y(s+\theta) - \bar{y}_s(\theta)|^p \leq [&2^{p-1}\beta + 3^{p-1}2^{\frac{3p}{2}-2}K^{\frac{p}{2}}(1 + H(p))(1 + \bar{H} + C)\triangle \\ &+ 2^{p-1}E|y(v) - \bar{y}(k_v\triangle)|^p. \end{aligned} \tag{3.10}$$

In what follows, we divide the following five cases to estimate the second term on the right-hand side of (3.10).

Case 1: $k_v \geq 0$ and $0 \leq v - k_v\triangle < \triangle$. By (2.8)

$$\begin{aligned}
&E|y(v) - \bar{y}(k_v\triangle)|^p \\
&\quad = E|f(\bar{y}_{k_v\triangle})(v - k_v\triangle) + g(\bar{y}_{k_v\triangle})(B(v) - B(k_v\triangle)) + h(\bar{y}_{k_v\triangle})(N(v) - N(k_v\triangle))|^p \\
&\quad \leq 3^{p-1}E|f(\bar{y}_{k_v\triangle})|^p(v - k_v\triangle)^p + 3^{p-1}E|g(\bar{y}_{k_v\triangle})|^p E|B(v) - B(k_v\triangle)|^p \\
&\quad + 3^{p-1}E|h(\bar{y}_{k_v\triangle})|^p E|N(v) - N(k_v\triangle)|^p.
\end{aligned}$$

Then, in the same way as (3.9) was done, we have for $\triangle \in (0,1)$

$$E|y(v) - \bar{y}(k_v\triangle)|^p \leq 3^{p-1}2^{\frac{p}{2}-1}K^{\frac{p}{2}}(1 + H(p))(1 + \bar{H} + C)\,\triangle\,.$$

Case 2: $k_v \geq 0$ and $\triangle \leq v - k_v\triangle < 2\triangle$. It then follows easily that

$$\begin{aligned}
E|y(v) - \bar{y}(k_v\triangle)|^p &= E|y(v) - \bar{y}((k_v+1)\triangle) + \bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)|^p \\
&\leq 2^{p-1}E|y(v) - \bar{y}((k_v+1)\triangle)|^p + 2^{p-1}E|\bar{y}((k_v+1)\triangle) - \bar{y}(k_v\triangle)|^p.
\end{aligned}$$

This, together with (3.9) and Case 1, leads to

$$E|y(v) - \bar{y}(k_v\triangle)|^p \leq 3^{p-1}2^{\frac{3p}{2}-1}K^{\frac{p}{2}}(1 + H(p))(1 + \bar{H} + C)\,\triangle\,.$$

Case 3: $k_v = -1$ and $0 \leq v - k_v\triangle \leq \triangle$. In this case, $-\triangle \leq v \leq 0$. We then have from (2.5) that

$$E|y(v) - \bar{y}(k_v\triangle)|^p \leq \beta\,\triangle\,.$$

Case 4: $k_v = -1$ and $\triangle \leq v - k_v\triangle < 2\triangle$. In such case, $0 \leq v < \triangle$. Case 1 and Case 2 can be used to estimate the term

$$\begin{aligned}
E|y(v) - \bar{y}(k_v\triangle)|^p &\leq 2^{p-1}E|y(v) - \xi(0)|^p + 2^{p-1}E|\xi(0) - \bar{y}((k_v\triangle)|^p \\
&\leq [2^{p-1}\beta + 3^{p-1}2^{\frac{3p}{2}-2}K^{\frac{p}{2}}(1 + H(p))(1 + \bar{H} + C)]\,\triangle\,.
\end{aligned}$$

Case 5: $k_v \leq -2$. In this case, $v < 0$. So, by (2.5)

$$E|y(v) - \bar{y}(k_v\triangle)|^p \leq 2\beta\,\triangle\,.$$

Combining case 1 to case 5, we therefore complete the proof. □

The following Theorem will tell us the error of the pth moment between the true solution and numerical solution under global Lipschitz condition.

Theorem 3.1. *Under the conditions* (2.3) *and* (2.5)*, for* $p \geq 2$*,*

$$E\left(\sup_{0\leq t\leq T}|x(t) - y(t)|^p\right) \leq \delta_1 L^{\frac{p}{2}} e^{\delta_2 L^{\frac{p}{2}}}\triangle, \tag{3.11}$$

where δ_1, δ_2 *are constants which are independent of* $\triangle$*.*

Proof. It is easy to see from (2.1) and (2.8) that for any $t_1 \in [0,T]$

$$
\begin{aligned}
E\left(\sup_{0\le t\le t_1}|x(t)-y(t)|^p\right) &\le 3^{p-1}E\left(\sup_{0\le t\le t_1}\left|\int_0^t f(x_{s^-})-f(\bar{y}_{s^-})ds\right|^p\right)\\
&+3^{p-1}E\left(\sup_{0\le t\le t_1}\left|\int_0^t g(x_{s^-})-g(\bar{y}_{s^-})dB(s)\right|^p\right)\\
&+3^{p-1}E\left(\sup_{0\le t\le t_1}\left|\int_0^t h(x_{s^-})-h(\bar{y}_{s^-})dN(s)\right|^p\right)\\
&:= I_1+I_2+I_3.
\end{aligned}
\tag{3.12}
$$

In the sequel, we estimate these terms respectively. By the Hölder inequality, (2.3) and Lemma 3.2,

$$
\begin{aligned}
I_1 &\le 3^{p-1}T^{p-1}\int_0^{t_1}E|f(x_s)-f(\bar{y}_s)|^p ds\\
&\le 6^{p-1}T^{p-1}\int_0^{t_1}E|f(x_s)-f(y_s)|^p ds+6^{p-1}T^{p-1}\int_0^{t_1}E|f(y_s)-f(\bar{y}_s)|^p ds\\
&\le 6^{p-1}T^{p-1}\int_0^{t_1}E\left(\int_{-\tau}^0|x(s+\theta)-y(s+\theta)|^2d\mu(\theta)\right)^{\frac{p}{2}}ds\\
&+6^{p-1}T^{p-1}\int_0^{t_1}E\left(\int_{-\tau}^0|y(s+\theta)-\bar{y}_s(\theta)|^2d\mu(\theta)\right)^{\frac{p}{2}}ds\\
&\le 6^{p-1}T^{p-1}L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)ds+6^{p-1}T^{p-1}L^{\frac{p-2}{2}}\int_0^{t_1}\int_{-\tau}^0 E|y(s+\theta)-\bar{y}_s(\theta)|^p d\mu(\theta)ds\\
&\le 6^{p-1}T^pL^{\frac{p}{2}}\gamma\triangle+6^{p-1}T^{p-1}L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)ds.
\end{aligned}
$$

Now, the Burkholder-Davis-Gundy inequality [8, Theorem 7.3, p40], (2.3) and Lemma 3.2 also give that, for some positive constant C_p,

$$
\begin{aligned}
I_2 &\le 3^{p-1}C_pE\left(\int_0^{t_1}|g(x_s)-g(\bar{y}_s)|^2ds\right)^{\frac{p}{2}}\\
&\le 3^{p-1}T^{\frac{p-2}{2}}C_p\int_0^{t_1}E|g(x_s)-g(\bar{y}_s)|^p ds\\
&\le 6^{p-1}T^{\frac{p-2}{2}}C_p\left[L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)ds+L^{\frac{p-2}{2}}\int_0^{t_1}\int_{-\tau}^0E|y(s+\theta)-\bar{y}_s(\theta)|^pd\mu(\theta)ds\right]\\
&\le 6^{p-1}\gamma T^{\frac{p}{2}}C_pL^{\frac{p}{2}}\triangle+6^{p-1}T^{\frac{p-2}{2}}C_pL^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)ds.
\end{aligned}
\tag{3.13}
$$

In the same way as (3.13) was done, together with the Burkholder-Davis-Gundy inequality

[10, Theorem 48, p193], we can deduce from (2.3) that, for some positive constant $\bar{C}_p$,

$$
\begin{aligned}
I_3 &\le 6^{p-1} E\left(\sup_{0\le t\le t_1}\left|\int_0^t h(x_{s^-})-h(\bar{y}_{s^-})d\tilde{N}(s)\right|^p+\lambda^p\sup_{0\le t\le t_1}\left|\int_0^t h(x_s)-h(\bar{y}_s)ds\right|^p\right)\\
&\le 6^{p-1}(\bar{C}_pT^{\frac{p-2}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^{p-1})\int_0^{t_1}E|h(x_s)-h(\bar{y}_s)|^pds\\
&\le 12^{p-1}(\bar{C}_pT^{\frac{p-2}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^{p-1})L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)ds\\
&+12^{p-1}(\bar{C}_pT^{\frac{p-2}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^{p-1})L^{\frac{p-2}{2}}\int_0^{t_1}\int_{-\tau}^0 E|y(s+\theta)-\bar{y}_s(\theta)|^pd\mu(\theta)ds\\
&\le 12^{p-1}\gamma(\bar{C}_pT^{\frac{p}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^p)L^{\frac{p}{2}}\triangle+12^{p-1}(\bar{C}_pT^{\frac{p-2}{2}}\lambda^{\frac{p}{2}}+\lambda^pT^{p-1})L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)ds.
\end{aligned}
$$

Therefore

$$
E\left(\sup_{0\le t\le t_1}|x(t)-y(t)|^p\right)\le\delta_1L^{\frac{p}{2}}\triangle+\delta_2L^{\frac{p}{2}}\int_0^{t_1}E\left(\sup_{0\le r\le s}|x(r)-y(r)|^p\right)ds,
$$

where $\delta_1=6^{p-1}\gamma T^{\frac{p}{2}}\left(T^{\frac{p}{2}}+C_p+2^{p-1}\bar{C}_p\lambda^{\frac{p}{2}}+\lambda^pT^{\frac{p}{2}}\right)$ and $\delta_2=6T^{\frac{p-2}{2}}\Big(T^{\frac{p}{2}}+C_p+2^{p-1}\bar{C}_p\lambda^{\frac{p}{2}}+\lambda^pT^{\frac{p}{2}}\Big)$. The desired assertion thus follows from the Gronwall inequality. □

Remark 3.1. *The result of Theorem 3.1 tells us*

$$
E\left(\sup_{0\le t\le T}|x(t)-y(t)|^2\right)\le\delta_3Le^{\delta_4L}\triangle, \tag{3.14}
$$

where δ_3,δ_4 are constants which are independent of $\triangle$ under the global Lipschitz condition (2.3). This means that the order of the mean-square convergence is $1/2$, while Eq. (3.11) tells us that the order of the pth moment convergence is $1/p$ ($p\ge2$). In other words, the lower moment has a better convergence rate for the SFDEs with jumps, whence it is best in practice to use the mean-square convergence. This is significantly different from the result on SFDEs without jumps. Letting $h\equiv0$ in (2.1), i.e. there is no jumps, we have already known that for $p\ge2$ (see [12])

$$
E\left(\sup_{0\le t\le T}|x(t)-y(t)|^p\right)\le\hat{C}_1\triangle^{p/2},
$$

where $\hat{C}_1$ is a constant independent of $\triangle$. This means that the order of the pth moment convergence is $1/2$ for all $p\ge2$. Why is there a significant difference? Actually, it is due to the following fact: all moments of the Poisson increments $N((k+1)\triangle)-N(k\triangle)$ have the same order of $\triangle$ (see(3.8)), while the moments of increments $\triangle B_k=B((k+1)\triangle)-B(k\triangle)$ have different orders, namely $E|\triangle B_k|^{2n}=O(\triangle^n)$ and $E|\triangle B_k|^{2n+1}=0$.

4 Rate of convergence under local Lipschitz condition

In this section, we shall discuss the rate of convergence of EM numerical solutions to (2.1) under the following *local Lipschitz condition.*

(H3) (Local Lipschitz condition) For each integer $j \geq 1$, there is a left-continuous nondecreasing function $\mu_j : [-\tau, 0] \to R_+$ such that

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq \int_{-\tau}^{0} |\varphi(\theta) - \psi(\theta)|^2 d\mu_j(\theta), \quad (4.1)$$

for those $\varphi, \psi \in \hat{D}([-\tau, 0]; R^n)$ with $\|\varphi\| \vee \|\psi\| \leq j$.

(H4) (Linear growth condition) Assume that there is a constant $h > 0$ such that, for $\varphi \in \hat{D}([-\tau, 0]; R^n)$,

$$|f(\varphi)|^2 \vee |g(\varphi)|^2 \vee |h(\varphi)|^2 \leq h(1 + \|\varphi\|^2). \quad (4.2)$$

Remark 4.1. *Under the conditions* (4.1) *and* (4.2)*, for any initial data* $\xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$*,* (2.1) *admits a unique solution* $x(t), t \in [0, T]$ *by using the standard truncation procedure (see [8, Theorem 3.4, p56]). Moreover,* (4.1) *implies for those* $\varphi, \psi \in \hat{D}([-\tau, 0]; R^n)$ *with* $\|\varphi\| \vee \|\psi\| \leq j$

$$|f(\varphi) - f(\psi)|^2 \vee |g(\varphi) - g(\psi)|^2 \vee |h(\varphi) - h(\psi)|^2 \leq L_j \|\varphi - \psi\|^2, \quad (4.3)$$

where $L_j = \mu_j(0) - \mu_j(-\tau)$*.*

Theorem 4.1. *Let conditions* (2.5), (4.1) *and* (4.2) *hold. If there exist positive constant* α *and* $\tilde{\varepsilon} \in (0, 1)$ *such that the local Lipschitz constant obeys*

$$L_j^{1+\tilde{\varepsilon}} \leq \alpha \log j, \quad (4.4)$$

then

$$E\left(\sup_{0 \leq t \leq T} |x(t) - y(t)|^2\right) = O(\triangle^{\frac{2}{2+\epsilon}}), \quad (4.5)$$

where $\epsilon \in (0, \tilde{\varepsilon})$ *is an arbitrarily fixed small positive number.*

Proof. Let $j \geq 1$ be an integer, and let $S_j = \{x \in R^n : |x| \leq j\}$. Define the projection $\pi_j : R^n \to S_j$ by

$$\pi_j(x) = \frac{j \wedge |x|}{|x|} x,$$

where we set $\pi_j(0) = 0$ as usual. It is easy to see that for all $x, y \in R^n$

$$|\pi_j(x) - \pi_j(y)| \leq |x - y|.$$

Define the operator $\bar{\pi}_j : \hat{D}([-\tau, 0]; R^n) \to \hat{D}([-\tau, 0]; R^n)$ by

$$\bar{\pi}_j(\varphi) = \{\pi_j(\varphi(\theta)) : -\tau \le \theta \le 0\}.$$

Clearly,

$$\|\bar{\pi}_j(\varphi)\| \le j, \qquad \forall \varphi \in \hat{D}([-\tau, 0]; R^n).$$

Define the truncation functions $f_j : \hat{D}([-\tau, 0]; R^n) \to R^n$, $g_j : \hat{D}([-\tau, 0]; R^n) \to R^{n\times m}$ and $h_j : \hat{D}([-\tau, 0]; R^n) \to R^n$ by

$$f_j(\varphi) = f(\bar{\pi}_j(\varphi)), \qquad g_j(\varphi) = g(\bar{\pi}_j(\varphi)), \qquad h_j(\varphi) = h(\bar{\pi}_j(\varphi)), \tag{4.6}$$

respectively. Then, by (4.1), for any $\varphi, \psi \in \hat{D}([-\tau, 0]; R^n)$,

$$\begin{aligned}
&|f_j(\varphi) - f_j(\psi)|^2 \vee |g_j(\varphi) - g_j(\psi)|^2 \vee |h_j(\varphi) - h_j(\psi)|^2 \\
&\le |f(\bar{\pi}_j(\varphi)) - f(\bar{\pi}_j(\psi))|^2 \vee |g(\bar{\pi}_j(\varphi)) - g(\bar{\pi}_j(\psi))|^2 \vee |h(\bar{\pi}_j(\varphi)) - h(\bar{\pi}_j(\psi))|^2 \\
&\le \int_{-\tau}^{0} |\pi_j(\varphi(\theta)) - \pi_j(\psi(\theta))|^2 d\mu_j(\theta) \\
&\le \int_{-\tau}^{0} |\varphi(\theta) - \psi(\theta)|^2 d\mu_j(\theta).
\end{aligned} \tag{4.7}$$

That is, f_j, g_j and h_j satisfy the global Lipschitz condition. For $t \in [0, T]$, let $x^j(t)$ be the solution to the following SFDE with jumps

$$dx^j(t) = f_j(x^j_{t-})dt + g_j(x^j_{t-})dB(t) + h_j(x^j_{t-})dN(t)$$

with the initial data $x^j_0 = \xi$ and $y^j(t)$ be the corresponding continuous-time EM solution with the stepsize $\triangle$. By Theorem 3.1 for any sufficiently small$\epsilon \in (0, \tilde{\varepsilon})$

$$E\left(\sup_{0\le t\le T} |x^j(t) - y^j(t)|^{2+\epsilon}\right) \le \delta_1 L_j^{1+\epsilon/2} e^{\delta_2 L_j^{1+\epsilon/2}} \triangle .$$

Furthermore, by (4.4) (here we assume $L_j \ge 1$ without any loss of generality),

$$E\left(\sup_{0\le t\le T} |x^j(t) - y^j(t)|^{2+\epsilon}\right) \le e^{(\delta_1+\delta_2)L_j^{1+\epsilon/2}} \triangle \le j^{\alpha(\delta_1+\delta_2)} \triangle . \tag{4.8}$$

Set

$$\hat{x}(T) = \sup_{0\le t\le T} |x(t)| \quad \textit{and} \quad \hat{y}(T) = \sup_{0\le t\le T} |y(t)|.$$

For any integer $j \ge 1$, define stopping time

$$\tau_j = T \wedge \inf\{t \in [0, T] : \|x^j_t\| \vee \|y^j_t\| \ge j\}.$$

It is easy to see that $\|x^j_{s-}\| \le j$ for any $0 \le s < \tau_j$. Then, combining (4.6) gives that for any $0 \le s < \tau_j$

$$f_j(x^j_{s-}) = f\left(\frac{\|x^j_{s-}\| \wedge j}{\|x^j_{s-}\|} x^j_{s-}\right) = f\left(\frac{\|x^j_{s-}\| \wedge (j+1)}{\|x^j_{s-}\|} x^j_{s-}\right) = f_{j+1}(x^j_{s-}) = f(x^j_{s-}).$$

Similarly,

$$g_j(x^j_{s^-}) = g_{j+1}(x^j_{s^-}) = g(x^j_{s^-}), \qquad h_j(x^j_{s^-}) = h_{j+1}(x^j_{s^-}) = h(x^j_{s^-}).$$

While on $0 \le t < \tau_j$

$$\begin{aligned}
x^j(t) &= \xi(0) + \int_0^t f_j(x^j_{s^-})ds + \int_0^t g_j(x^j_{s^-})dB(s) + \int_0^t h_j(x^j_{s^-})dN(s) \\
&= \xi(0) + \int_0^t f_{j+1}(x^j_{s^-})ds + \int_0^t g_{j+1}(x^j_{s^-})dB(s) + \int_0^t h_{j+1}(x^j_{s^-})dN(s) \\
&= \xi(0) + \int_0^t f(x^j_{s^-})ds + \int_0^t g(x^j_{s^-})dB(s) + \int_0^t h(x^j_{s^-})dN(s).
\end{aligned}$$

Consequently, we must have that

$$x(t) = x^j(t) = x^{j+1}(t)$$

on $0 \le t < \tau_j$. Likewise, we can also derive that

$$y(t) = y^j(t) = y^{j+1}(t)$$

for $0 \le t < \tau_j$. These imply that τ_j is non-decreasing and, by Lemma 3.1, $\lim_{j\to\infty} \tau_j = T$ a.s. Let $\tau_0 = 0$ and compute, for $t \in [0, T]$,

$$\begin{aligned}
|x(t) - y(t)|^2 &= \sum_{j=1}^{\infty} |x(t) - y(t)|^2 I_{[\tau_{j-1} \le t < \tau_j)} \\
&= \sum_{j=1}^{\infty} |x^j(t) - y^j(t)|^2 I_{[\tau_{j-1} \le t < \tau_j)} \\
&\le \sum_{j=1}^{\infty} |x^j(t) - y^j(t)|^2 I_{[j-1 \le \hat{x}(T) \vee \hat{y}(T)]}.
\end{aligned}$$

Therefore, by the Hölder inequality

$$\begin{aligned}
&E\left(\sup_{0\le t\le T} |x(t) - y(t)|^2\right) \\
&\le \sum_{j=1}^{\infty} \left(E\left(\sup_{0\le t\le T} |x^j(t) - y^j(t)|^{2+\epsilon}\right)\right)^{\frac{2}{2+\epsilon}} \left(EI_{[j-1 \le \hat{x}(T)\vee\hat{y}(T)]}\right)^{\frac{\epsilon}{2+\epsilon}} \\
&\le \sum_{j=1}^{\infty} \left(E\left(\sup_{0\le t\le T} |x^j(t) - y^j(t)|^{2+\epsilon}\right)\right)^{\frac{2}{2+\epsilon}} [P(j-1 \le \hat{x}(T) \vee \hat{y}(T))]^{\frac{\epsilon}{2+\epsilon}}.
\end{aligned} \tag{4.9}$$

On the other hand, for any $q \ge 2$, we obtain from Lemma 3.1

$$P(j-1 \le \hat{x}(T) \vee \hat{y}(T)) \le \frac{E|\hat{x}(T)|^q + E|\hat{y}(T)|^q}{(\frac{j}{2})^q} \le \frac{2H(q)}{(\frac{j}{2})^q} \tag{4.10}$$

with $j \geq 2$. Substituting (4.8) and (4.10) into (4.9), one has

$$E\left(\sup_{0\leq t\leq T}|x(t)-y(t)|^2\right) \leq \left(1+2^{\frac{q\epsilon}{2+\epsilon}}(2H(q))^{\frac{\epsilon}{2+\epsilon}}\sum_{j=2}^{\infty} j^{\frac{2\alpha(\delta_1+\delta_2)-q\epsilon}{2+\epsilon}}\right)\triangle^{\frac{2}{2+\epsilon}}. \tag{4.11}$$

For any fixed $\epsilon > 0$ letting q be sufficiently large for

$$q \geq \frac{\alpha(\delta_1+\delta_2)+2(2+\epsilon)}{\epsilon},$$

we see that the right-hand side of (4.11) is convergent, whence the desired assertion (4.5) follows. $\square$

Remark 4.2. *Under the local Lipschitz condition, Mao [9] showed the strong convergence of the numerical solutions to SFDEs without jumps, and the rate of convergence was revealed under the global Lipschitz condition. In the present paper, under the local Lipschitz condition, we reveal the rate of convergence for the numerical solutions to SFDEs with jumps. The rate of convergence for jump processes* (2.1) *we revealed here is* $1/(2+\epsilon)$ *(closed to* $1/2$*) under the logarithm growth condition* (4.4)*. This is different from the rate of convergence for the diffusion processes (without jumps) which was studied in [12], where it was shown that the rate of convergence is still* $1/2$ *under the logarithm growth condition. The reason for such a difference has already been pointed out in Remark 3.1.*

5 Appendix: an existence-and-uniqueness theorem

To make our paper self-contained, in this section we shall discuss the existence and uniqueness of solutions to (2.1) under the assumption $(H1)$.

Theorem 5.1. *Under the conditions* (2.2)*, there exists a unique solution* $x(t), t \in [0,T]$*, to* (2.1) *for any initial data* $\xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$*.*

Proof. Since our proof is an application of the proof for the case without jumps in [8, Theorem 2.2, p150], we here give only a sketch for the proof of jump case.

Uniqueness. Let $x(t)$ and $\bar{x}(t)$ be two solutions to (2.1) on $[0,T]$. Noting from (2.1) that

$$x(t)-\bar{x}(t) = \int_0^t [f(x_{s^-})-f(\bar{x}_{s^-})]ds + \int_0^t [g(x_{s^-})-g(\bar{x}_{s^-})]dB(s) + \int_0^t [h(x_{s^-})-h(\bar{x}_{s^-})]dN(s)$$

and $\tilde{N}(t) = N(t) - \lambda t$ is a martingale measure for $t \in [0,T]$, along with (2.3) we have

$$E\left(\sup_{0\leq t\leq T}|x(t)-\bar{x}(t)|^2\right) \leq 3L(T+4+8\lambda+2\lambda^2 T)\int_0^T E\left(\sup_{0\leq r\leq s}|x(r)-\bar{x}(r)|^2\right)ds.$$

By the Gronwall inequality

$$E\left(\sup_{0\leq t\leq T}|x(t)-\bar{x}(t)|^2\right) = 0,$$

which implies that $x(t) = \bar{x}(t)$ for $t \in [0, T]$ almost surely. The uniqueness has been proved.

Existence. Define $x_0^0 = \xi$ and $x^0(t) = \xi(0)$ for $0 \le t \le T$. For each $n = 1, 2, \cdots$, set $x_0^n = \xi$ and define, by the Picard iterations,

$$x^n(t) = \xi(0) + \int_0^t f(x_{s^-}^{n-1})ds + \int_0^t g(x_{s^-}^{n-1})dB(s) + \int_0^t h(x_{s^-}^{n-1})dN(s) \tag{5.1}$$

for $t \in [0, T]$. It also follows from (5.1) that for any integer $k \ge 1$

$$E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le \bar{c}_1 + \bar{c}_2 \int_0^T E\left(\sup_{0\le r\le s} |x^{n-1}(r)|^2\right) ds,$$

where $\bar{c}_1 = 4[E\|\xi\|^2 + K(T^2+4T) + (8\lambda + 2\lambda^2 T)T] + \bar{c}_2 T E\|\xi\|^2$ and $\bar{c}_2 = 4K[T+4+8\lambda+2\lambda^2 T]$. This further implies that

$$\max_{1\le n\le k} E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le \bar{c}_1 + \bar{c}_2 \int_0^T \max_{1\le n\le k} E\left(\sup_{0\le s\le t} |x^{n-1}(s)|^2\right) dt.$$

Observing

$$\begin{aligned}
&\max_{1\le n\le k} E\left(\sup_{0\le s\le t} |x^{n-1}(s)|^2\right)\\
&= \max\left\{E|\xi(0)|^2, E\left(\sup_{0\le s\le t} |x^1(s)|^2\right), \cdots, E\left(\sup_{0\le s\le t} |x^{k-1}(s)|^2\right)\right\}\\
&\le \max\left\{E\|\xi\|^2, E\left(\sup_{0\le s\le t} |x^1(s)|^2\right), \cdots, E\left(\sup_{0\le s\le t} |x^{k-1}(s)|^2\right), E\left(\sup_{0\le s\le t} |x^k(s)|^2\right)\right\}\\
&\le E\|\xi\|^2 + \max_{1\le n\le k} E\left(\sup_{0\le s\le t} |x^n(s)|^2\right),
\end{aligned}$$

we hence deduce that

$$\max_{1\le n\le k} E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le \bar{c}_1 + \bar{c}_2 T E\|\xi\|^2 + \bar{c}_2 \int_0^T \max_{1\le n\le k} E\left(\sup_{0\le s\le t} |x^n(s)|^2\right) dt.$$

The Gronwall inequality implies

$$\max_{1\le n\le k} E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le (\bar{c}_1 + \bar{c}_2 T E\|\xi\|^2)e^{\bar{c}_2 T}.$$

Since k is arbitrary, we must have for $n \ge 1$

$$E\left(\sup_{0\le t\le T} |x^n(t)|^2\right) \le (\bar{c}_1 + \bar{c}_2 T E\|\xi\|^2)e^{\bar{c}_2 T}.$$

Next, by (5.1)

$$\begin{aligned}
E\left(\sup_{0\le t\le T} |x^1(t) - x^0(t)|^2\right) &\le 3K(T + 4 + 8\lambda + 2\lambda^2 T) \int_0^T (1 + E\|x_s^0\|^2)ds\\
&\le 3KT(T + 4 + 8\lambda + 2\lambda^2 T)(1 + E\|\xi\|^2) := \bar{C}.
\end{aligned} \tag{5.2}$$

We now claim that for $n \geq 0$

$$E\left(\sup_{0\leq s\leq t}|x^{n+1}(s)-x^n(s)|^2\right) \leq \frac{\bar{C}M^n t^n}{n!}, \qquad 0\leq t\leq T, \tag{5.3}$$

where $M = 3K(T+4+8\lambda+2\lambda^2 T)$. We shall show this by induction. In view of (5.2) we see that (5.3) holds whenever $n=0$. Now, assume that (5.3) holds for some $n\geq 0$. Then,

$$\begin{aligned}
E\left(\sup_{0\leq s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2\right) &\leq M\int_0^t E\|x_s^{n+1}-x_s^n\|^2 ds \\
&\leq M\int_0^t E\left(\sup_{0\leq r\leq s}|x^{n+1}(r)-x^n(r)|^2\right) ds \\
&\leq M\int_0^t \frac{\bar{C}M^n s^n}{n!} ds = \frac{\bar{C}M^{n+1}t^{n+1}}{(n+1)!}.
\end{aligned}$$

Following the proof of [8, Theorem 3.1, p55], we can show that for almost all $\omega\in\Omega$ there exists a positive integer $n_0 = n_0(\omega)$ such that

$$\sup_{0\leq s\leq T}|x^{n+1}(s)-x^n(s)| \leq \frac{1}{2^n} \quad \text{whenever } n\geq n_0(\omega). \tag{5.4}$$

This implies $\{x^n(\cdot)\}_{n\geq 1}$ is a Cauchy sequence under $\sup|\cdot|$. However, since our space $D([0,T];R^n)$ is not a complete space under $\sup|\cdot|$, we do not know whether $\{x^n(\cdot)\}_{n\geq 1}$ has a limit in $D([0,T];R^n)$. In order for $D([0,T];R^n)$ is complete, we need to define the following metric (see [1, Chapter 3]). Let Λ denote the class of strictly increasing, continuous mapping of $[0,T]$ onto itself and

$$\Lambda_\epsilon^* = \left\{\lambda\in\Lambda : \sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| \leq \epsilon\right\},$$

define

$$d(\xi,\zeta) = \inf\{\epsilon>0 : \exists\lambda\in\Lambda_\epsilon^* \text{ such that } \sup_{t\in[0,T]}|\xi(t)-\zeta(\lambda(t))|\leq\epsilon\}.$$

$d(\cdot,\cdot)$ is called the Skorohod metric, by [1, Theorem 14.2, p115] we know that $D([0,T];R^n)$ is complete in the metric d. Taking $\lambda(t)=t$, we can see $\{x^n(\cdot)\}_{n\geq 1}$ is a Cauchy sequence under d. Therefore there exists unique $x(t), t\in[0,T]\in D([0,T];R^n)$ such that $d(x^n(\cdot),x(\cdot))\to 0$ as $n\to\infty$. Taking the limit in (5.1), we then can show that $x(t)$ is the solution of (2.1). $\square$

References

[1] Billingsley, P., *Convergence of Probability Measures*, John Wiley and Sons, 1968.

[2] Cont, R. and Tankov, P., *Financial Modelling With Jump Processes*, Chapman and Hall/CRC, Florida, 2004.

[3] Gukhal, C. R., *The compound option approach to American options on jump-diffusions*, J. Econom. Dynam. Control, 28 (2004) 2055-2074.

[4] Higham, D.J. and Kloeden, P.E., *Numerical methods for nonlinear stochastic differential equations with jumps*, Numer. Math., 101 (2005) 101-119.

[5] Higham, D.J. and Kloeden, P.E., *Strong convergence rates for backward Euler on a class of nonlinear jump-diffusion problems*, J. Comput. Appl. Math., 205 (2007) 949-956.

[6] Hale, J. K. and Lunel, S. M. V., *Introduction to Functional Differential Equations,* Springer, New York, 1993.

[7] Kolmanovskii, V. B. and Nosov, V. R., *Stability and Periodic Modes of Control Systems with After effect*, Nauka: Moscow, 1981.

[8] Mao, X., *Stochastic Differential Equations and Applications*, Horwood, 1997.

[9] Mao, X., *Numerical solutions of stochastic functional differential equations*, LMSJ. Comput. Math., 6 (2003) 141-161.

[10] Protter, P. E., *Stochastic Integration and Differential Equations*, second edition, Springer-Verlag, New York, 2004.

[11] Sobczyk, K., *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer Academic, Dordrecht, 1991.

[12] Yuan, C. and Mao, X., *A Note on the rate of convergence of the Euler-Maruyama method for scholastic differential equations*, Stoch. Anal. Appl., 26 (2008) 325-333.